When Fish Moonwalk

Alexandre Munnier, Thomas Chambrion

When Fish Moonwalk

Thomas Chambrion Alexandre Munnier

***Abstract*— In this paper we study some issues relating to the general problem of locomotion by shape-changes in a perfect fluid. Our results are two fold. First we introduce a rigorous model for a weighted self-propelled swimming body - one specificity of this model being that the number of the body's deformations degrees of freedom is infinite. The dynamic of the coupled system fluid-body is driven by the so-called Euler-Lagrange equations: a system of ODEs allowing us to compute the *rigid motion* of the body with respect to its prescribed shape-changes. Second, we prove controllability results for this model using powerful tools of geometric control theory. For instance, we show that the body can follow (approximately) any prescribed trajectory while undergoing (approximately) any prescribed shape-changes (this surprising phenomenon will be called *Moonwalking*). Most of our theoretical results are illustrated by numerical simulations.**

I. INTRODUCTION

A. State of the art in biomechanics-swimming

In the last decade, serious efforts have been done by mathematicians to better understand the dynamics of swimming in a fluid ([22], [12], [6], [24], [21], [19], [4]). Some models ([23], [25], [14]) incorporate artificial vortices. If we do not neglect the viscosity effects, the relevant model consists of the non stationary Navier-Stokes equations for the fluid coupled with Newton's laws for the fish-like swimming object ([2], [13], [7], [18]). However, contrary to some commonly-held beliefs, the forces and momenta applied to the fish body by shed vortices are not solely responsible for the net locomotion and most of the numerous articles by mathematicians studying fish locomotion address the case of a potential flow which is, by definition, vortex-free ([9], [10], [8], [15], [16], [17]). It is also the point of view we have chosen in this paper.
Although crucial for the design of autonomous underwater vehicles, results on control or motion planning for this kind of problems are very few; most of them deal specifically with articulated bodies, as in [1] (in a viscous fluid) or in [14] and [15] (in a potential flow).

B. Main results

The shape-changing body (sometimes called *amoeba* for its similarity with this single-celled animal) we consider in this paper is inspired by that of Shapere and Wilczek [20] and further discussed in [5]. However, in our model the mass and the changing inertia momentum are both taken into account. The shape-changes are prescribed as functions of time and used as *controls* to propel and steer the swimming animal. The Euler-Lagrange equations are obtained following the method described in [8], [16], [17], adapted here to the infinite dimensional case.

The contribution of this paper is two fold:

First, we provide a physically-coherent mathematically well-posed model for a shape-changing swimming organism in a 2D perfect fluid with potential flow.

Second, we prove advanced approximate controllability properties for this model (namely, any displacement can approximately be achieved with approximately any shape-change).

Owning to the lack of place, the proofs of some technical results will be omitted. They can be found in [3] available at `http://hal.archives-ouvertes.fr/hal-00422429/fr/`

Both authors are with Institut Elie Cartan UMR 7502, Nancy-Université, CNRS, INRIA, B.P. 239, F-54506 Vandoeuvre-lès-Nancy Cedex, France, and INRIA Lorraine, Projet CORIDA. Authors partially supported by ANRs CISIFS, GAOS, GCM and CPER AOC.
thomas.chambrion@iecn.u-nancy.fr
alexandre.munnier@iecn.u-nancy.fr

II. MODELING

The modeling requires consideration of a *physical space* and a *computational space*. Both are identified either with $\mathbf{R}^2$ or with the complex field $\mathbf{C}$.

The computational space is endowed with a frame $(\mathbf{E}_1, \mathbf{E}_2)$, D stands for the open unitary disk and $\Omega := \mathbf{C}\setminus\bar{D}$.

We introduce two frames in the *physical space* (identified with $\mathbf{R}^2$): a Galilean fixed one $(\mathbf{e}_1, \mathbf{e}_2)$ and a moving one $(\mathbf{e}_1^*, \mathbf{e}_2^*)$, whose origin coincides at any time with the center-of-mass of the swimming body. Therefore, the center-of-mass of the body has coordinates $\mathbf{r} := (r_1, r_2)^T$ in $(\mathbf{e}_1, \mathbf{e}_2)$ and $(0,0)^T$ in $(\mathbf{e}_1^*, \mathbf{e}_2^*)$. More generally, the amounts are denoted with asterisks when expressed in the moving frame: the domain occupied by the amoeba is $\mathcal{A}$ in $(\mathbf{e}_1, \mathbf{e}_2)$ and $\mathcal{A}^*$ in $(\mathbf{e}_1, \mathbf{e}_2)$ while $\mathcal{F} := \mathbf{R}^2 \setminus \bar{\mathcal{A}}$ and $\mathcal{F}^* := \mathbf{R}^2 \setminus \bar{\mathcal{A}}^*$ stand for the domain of the fluid.

A. Shape-changes

Let $\mathcal{S}$ be the Banach space consisting of the complex sequences $\mathbf{c} := (c_k)_{k\geq 1}$, $c_k = a_k + ib_k$, $a_k, b_k \in \mathbf{R}$ such that $\|\mathbf{c}\|_{\mathcal{S}} := \sum_{k\geq 1} k(|a_k| + |b_k|) < +\infty$. The unitary open ball of $\mathcal{S}$ is denoted by $\mathcal{B}$ and for any $N \in \mathbf{N}$ ($N \geq 1$), $\mathcal{S}_N$ is the N-dimensional subspace of $\mathcal{S}$ for which $c_k = 0$ if $k > N$. We can easily verify that $\mathcal{S} \subset \mathcal{T}$, where $\mathcal{T}$ is the set of the complex sequences such that $\|\mathbf{c}\|_{\mathcal{T}} := \left(\sum_{k\geq 1} k(|a_k|^2 + |b_k|^2)\right)^{1/2} < +\infty$.

The shape-changes of the body are described by means of a set of diffeomorphisms $\chi(\mathbf{c})$ depending on $\mathbf{c} \in \mathcal{B}$ ($\mathbf{c}$ will be called in the sequel the *shape* or *control* variable). For any $\mathbf{c} \in \mathcal{B}$, $\chi(\mathbf{c})$ maps $\bar{D}$ (the closed unitary disk of the computational space) onto $\bar{\mathcal{A}}^*$. It is defined in complex notation by:

$$\chi(\mathbf{c})(z) := z + \sum_{k\geq 1} c_k \bar{z}^k, \quad (z \in \bar{D}), \tag{1}$$

where $\bar{z} = z_1 - iz_2$ is the complex conjugate of $z = z_1 + iz_2$. Hence, we have $\bar{\mathcal{A}}^* := \chi(\mathbf{c})(\bar{D})$ and $\mathcal{A}^*$ does depend on the shape variable. We introduce likewise the function $\phi(\mathbf{c})$ that maps $\bar{\Omega}$ onto $\bar{\mathcal{F}}^*$. It is defined for all $\mathbf{c} \in \mathcal{B}$ by:

$$\phi(\mathbf{c})(z) := z + \sum_{k\geq 1} c_k z^{-k}, \quad (z \in \bar{\Omega}). \tag{2}$$

Proposition 1: For all $\mathbf{c} \in \mathcal{B}$, $\chi(\mathbf{c}) : \bar{D} \to \bar{\mathcal{A}}^*$ and $\phi(\mathbf{c}) : \bar{\Omega} \to \bar{\mathcal{F}}^*$ are both well-defined (the series in (1) and (2) converge for all z) and invertible. Further, $\phi(\mathbf{c})|_D$ is continuously differentiable, $\chi(\mathbf{c})|_\Omega$ is a conformal mapping and $\chi(\mathbf{c})|_{\partial D} = \phi(\mathbf{c})|_{\partial\Omega}$.

Within this model, the given of the shape-changes is nothing but the given of a function of time $t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{B}$ ($T > 0$). We assume that this function is continuous and piecewise $\mathcal{C}^1$. We denote $\dot{\mathbf{c}} = (\dot{c}_k)_{k\geq 1}$ its time derivative, $\dot{c}_k := \dot{a}_k + i\dot{b}_k$ ($k \geq 1$) and for all $z \in D$, $\dot{\chi}(\mathbf{c})(z) := \sum_{k\geq 1} \dot{c}_k \bar{z}^k$.

B. Rigid motion, rigid velocity

The rigid motion of the amoeba is described by elements $\mathbf{q} := (\mathbf{r}, \theta)$ of $\mathcal{Q} := \mathbf{R}^2 \times \mathbf{R}/2\pi$ where $\mathbf{r} \in \mathbf{R}^2$ is a vector giving the position of the center-of-mass of the body and θ and angle giving its orientation with respect to $(\mathbf{e}_1, \mathbf{e}_2)$. If we denote by $R(\theta) \in \mathrm{SO}(2)$ the rotation matrix of angle θ, then $R(\theta)\mathbf{e}_j = \mathbf{e}_j^*$ ($j = 1, 2$).

Consider a smooth function $t \in \mathbf{R} \mapsto \mathbf{q}(t) := (\mathbf{r}(t), \theta(t)) \in \mathcal{Q}$ and denote by $\dot{\mathbf{q}} := (\dot{\mathbf{r}}, \omega) \in \mathbf{R}^3$ the time derivative of $\mathbf{q}$. The *Eulerian* velocity of a point undergoing a rigid motion is $\mathbf{v}_r = \omega(x - \mathbf{r})^\perp + \dot{\mathbf{r}}$. It can also be expressed in the moving frame $(\mathbf{e}_1^*, \mathbf{e}_2^*)$: $\mathbf{v}_r^* = \omega(x^*)^\perp + \dot{\mathbf{r}}^*$ where $\dot{\mathbf{r}}^* := R^T\dot{\mathbf{r}}$. This leads us to introduce the additional notation $\dot{\mathbf{q}}^* := (\dot{\mathbf{r}}^*, \omega)^T$.

C. Physical quantities

Let $\rho_0 > 0$ be a given constant seen as a density in $\bar{D}$. The conservation-of-density principle leads to the following expression for the densities ρ^* in $\mathcal{A}^*$ and ρ in $\mathcal{A}$:

$$\rho^*(x^*) = \rho_0|D\chi(\mathbf{c})(z)|^{-1}, \; (x^* = \chi(\mathbf{c})(z), \, z \in \bar{D}), \tag{3a}$$
$$\rho(x) = \rho^*(x^*), \qquad (x = R(\theta)x^* + \mathbf{r}, \, x^* \in \mathcal{A}^*), \tag{3b}$$

where $D\chi(\mathbf{c})(z)$ is the Jacobian matrix of $\chi(\mathbf{c})$ seen as a mapping from $\mathbf{R}^2$ into $\mathbf{R}^2$. We define next the mass elements in D, $\mathcal{A}^*$ and $\mathcal{A}$ respectively by $\mathrm{d}m_0 := \rho_0 \mathrm{d}z$, $\mathrm{d}m^* := \rho^*(x^*)\mathrm{d}x^*$ and $\mathrm{d}m := \rho(x)\mathrm{d}x$.

From identities (3) we deduce, after some algebra, that for all $\mathbf{c} \in \mathcal{B}$ the volume $\mathrm{Vol}(\mathcal{A})$ and the mass m of the amoeba are given respectively by:

$$\mathrm{Vol}(\mathcal{A}) = \pi\left(1 - \|\mathbf{c}\|_{\mathcal{T}}^2\right) \quad \text{and} \quad m = \pi\rho_0. \tag{4}$$

Since the fluid is assumed to be incompressible and we are dealing with a 2D model, its volume is always preserved (this is not true with a 3D model as explained in [17]) and we draw the same conclusion for the volume of the amoeba. We deduce that, for a shape function $t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{B}$ to be *physically allowable*, the quantity $\|\mathbf{c}(t)\|_{\mathcal{T}}$ has to be independent of time. We denote it $\mu := \|\mathbf{c}(0)\|_{\mathcal{T}}$. If the swimming body is assumed to be neutrally buoyant then $\rho_f(1 - \mu^2) = \rho_0$ where $\rho_f > 0$ stands for the given constant density of the fluid. We can also compute the inertia momentum in terms of the control variable:

$$I(\mathbf{c}) := \int_{\mathcal{A}^*} |x^*|^2 \, \mathrm{d}m^* = \pi\rho_0 \left[\frac{1}{2} + \sum_{k\geq 1} \frac{1}{k+1}|c_k|^2\right]. \tag{5}$$

D. Self-propelled motion, allowable control

The motion is said to be *self-propelled* when the shape-changes result from the work of internal forces and torques only. Such deformations are characterized by the fact that, in the absence of fluid, the linear and angular momenta of the amoeba are preserved:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{A}^*} x^* \mathrm{d}m^* &= \int_D \dot{\chi}(\mathbf{c})(z)\mathrm{d}m_0 \\ &= \sum_{k\geq 1} k(\dot{a}_k a_k + \dot{b}_k b_k) = 0,\end{aligned} \tag{6}$$

$$\int_D \dot{\chi}(\mathbf{c}) \cdot \chi(\mathbf{c})^\perp \mathrm{d}m_0 = \sum_{k\geq 1} \frac{1}{k+1}(\dot{b}_k a_k - \dot{a}_k b_k) = 0, \tag{7}$$

where we have used formula (3). According to the definition (1) of $\chi(\mathbf{c})$, the condition (6) is actually satisfied for any $\mathbf{c} \in \mathcal{B}$. We are now in a position to define precisely what is a *physically allowable control function*:

Definition 1 (Physically allowable control): Any continuous, piecewise $\mathcal{C}^1$ function $t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{B}$ for which (i) there exists $\mu \in]0, 1[$ such that $\|\mathbf{c}(t)\|_{\mathcal{T}} = \mu$ for all $t \in]0, T[$ and (ii) identity (7) holds for all t where $\dot{\mathbf{c}}$ exists, is said to be physically allowable.

E. Eulerian and convective velocities

According to the relation $x = \mathbf{r} + R(\theta)\chi(\mathbf{c})(z)$ ($z \in D$) and the results of Section II-B, the *Eulerian* and *convective* velocities of a material point of the body (whith coordinates x in $(\mathbf{e}_1, \mathbf{e}_2)$ and x^* in $(\mathbf{e}_1^*, \mathbf{e}_2^*)$) are:

$$\mathbf{v} = (\omega\,(x - \mathbf{r})^\perp + \dot{\mathbf{r}}) + R(\theta)\dot{\chi}(\mathbf{c})(\chi(\mathbf{c})^{-1}(g^{-1}x)), \tag{8a}$$
$$\mathbf{v}^* = (\omega\, x^{*\perp} + \dot{\mathbf{r}}^*) + \dot{\chi}(\mathbf{c})(\chi(\mathbf{c})^{-1}(x^*)). \tag{8b}$$

Notice that $\mathbf{v}$ is given in the fixed basis $(\mathbf{e}_1, \mathbf{e}_2)$ whereas $\mathbf{v}^*$ is given in the body-attached basis $(\mathbf{e}_1^*, \mathbf{e}_2^*)$.

F. Potential flow

The fluid is assumed to be incompressible and inviscid with constant density $\rho_f > 0$. Its element of mass is $\mathrm{d}m_f^* := \rho_f \mathrm{d}x^*$ in $\mathcal{F}^*$ and $\mathrm{d}m_f^0 := \rho_f \mathrm{d}z$ in Ω. The Eulerian velocity of the fluid in $(\mathbf{e}_1^*, \mathbf{e}_2^*)$, denoted by $\mathbf{u}^*$ is equal to the gradient of a *potential function* φ i.e. $\mathbf{u}^* = \nabla\varphi$ in $\mathcal{F}^*$. The incompressibility of the fluid entails that $\nabla \cdot \mathbf{u} = 0$ and hence that

$$\Delta\varphi = 0 \text{ in } \mathcal{F}^*. \tag{9a}$$

The classical *non-penetrating* or *slip* boundary condition for inviscid fluids leads to $\mathbf{u}^* \cdot n = \mathbf{v}^* \cdot n$ on $\partial\mathcal{F}^*$ where n stands here (and subsequently) for the unitary normal to $\partial\mathcal{A}^* = \partial\mathcal{F}^*$ directed toward the interior of $\mathcal{A}^*$. These conditions yield Neumann boundary conditions for the potential function:

$$\partial_n\varphi = \mathbf{v}^* \cdot n \text{ on } \partial\mathcal{F}^*. \tag{9b}$$

The boundary value problem (9) admits a weak (or variational) solution in the weighted Sobolev space $H^1_N(\mathcal{F}^*) := \{u \in \mathcal{D}'(\mathcal{F}^*) : u/[\sqrt{|x|^2+1}\log(2+|x|^2)] \in L^2(\mathcal{F}^*),\ \partial_{x_j}u \in L^2(\mathcal{F}^*),\ j=1,2\}$ where $\mathcal{D}'(\mathcal{F}^*)$ is the space of the distributions in $\mathcal{F}^*$. This solution is unique up to an additive constant.

Note that the potential function does depend on both $\dot{\mathbf{c}}$ (linearly through its boundary data) and $\mathbf{c}$ (through the domain $\mathcal{F}^*$).

G. Lagrangian function of the system fluid-body

We disregard the effects of gravity so the Lagrangian function L of our system reduces to: $L = K^b + K^f$ where K^b is the kinetic energy of the body of K^f the kinetic energy of the fluid. They read respectively:

$$K^b := \frac{1}{2}\int_{\mathcal{A}^*}|\mathbf{v}^*|^2\mathrm{d}m \text{ and } K^f := \frac{1}{2}\int_{\mathcal{F}^*}|\mathbf{u}^*|^2\mathrm{d}m_f. \tag{10}$$

Replacing $\mathbf{v}^*$ and $\mathbf{u}^*$ by their expressions and taking into account constraints (6) and (7), it turns out that the Lagrangian is a function of $\dot{\mathbf{c}}$, $\mathbf{c}$ and $\dot{\mathbf{q}}^*$. More precisely, for any fixed $\mathbf{c} \in \mathcal{B}$, $L(\mathbf{c})$ is a quadratic form in $(\dot{\mathbf{q}}^*, \dot{\mathbf{c}})$. It is worth remarking that it does not depend on $\mathbf{r}$ and θ due to the isotropy of our model with respect to the position and orientation of the body in the fluid.

H. Potential decomposition

Kirchhoff's law states that the potential function can be decomposed into a linear combination of elementary potentials, each one being assocated with a degree of freedom of the system (which are here: the translations of the body along $\mathbf{e}_j$ $(j = 1, 2)$, the rotation and all of the elementary shape-changes relating to the variables c_k $(k \geq 1)$). This law is classical when the number of degrees of freedom is finite but requires some adjustment to be applied to our infinite dimensional model.

Proposition 2: For any allowable control in the sense of Definition 1, we have the decompositions in $H^1_N(\mathcal{F}^*)$:

$$\begin{aligned}\varphi &= \varphi^r + \varphi^d,\\ \varphi^r &= \dot{r}_1^*\varphi_1^r + \dot{r}_2^*\varphi_2^r + \omega\varphi_3^r,\\ \varphi^d &= \sum_{k\geq 1}\dot{a}_k\varphi_k^a + \dot{b}_k\varphi_k^b,\end{aligned}$$

where the elementary potentials φ_j^r $(j = 1, 2)$ are associated with translations, φ_3^r with the rotation and φ_k^a and φ_k^b with the elementary shape-changes.

Notice once again that all of the potentials depend on $\mathbf{c}$ since the domain $\mathcal{F}^*$ does. Composing them with the conformal mapping $\phi(\mathbf{c})$, we obtain functions defined as solutions of Laplace's equations in the fixed domain Ω and for which the dependence in $\mathbf{c}$ is easier to analyse. This dependence has the same regularity as the dependence of the boundary data in $L^2(\partial\Omega)$ with respect to $\mathbf{c} \in \mathcal{B}$, that is, polynomial. Let us make clear this term: a mapping $f : E \to F$ between two Banach spaces is polynomial if there exist an integer p (the degree of the polynomial), a vector $a_0 \in F$ and $p-1$ mappings $\gamma_k : E \to F$ $(k = 1, ..., p)$ such that γ_k is k-linear and continuous and $f(\mathbf{e}) = \gamma_0 + \sum_{k\geq 1}\langle\gamma_k, \mathbf{e}, \ldots, \mathbf{e}\rangle$ for all $\mathbf{e} \in E$.

I. Mass matrices

By analogy with the classical definition (see [11]), we define for all $\mathbf{c} \in \mathcal{B}$ the mass matrix $\mathbb{M}(\mathbf{c})$ of our infinite dimensional system as being the polarization of the Lagrangian function $L(\mathbf{c})$ seen as a quadratic form in $(\dot{\mathbf{q}}^*, \dot{\mathbf{c}})$. We use the following notation, to emphasize the linearity with respect to some variables: $L(\mathbf{c}, \dot{\mathbf{c}}, \dot{\mathbf{q}}^*) = (1/2)\langle\mathbb{M}(\mathbf{c}), (\dot{\mathbf{c}}, \dot{\mathbf{q}}^*), (\dot{\mathbf{c}}, \dot{\mathbf{q}}^*)\rangle$. We can expand this formula and define the sub-mass matrices $\mathbb{M}^r(\mathbf{c})$, $\mathbb{N}(\mathbf{c})$ and $\mathbb{M}^d(\mathbf{c})$ by

$$\begin{aligned}\langle\mathbb{M}(\mathbf{c}), (\dot{\mathbf{c}}, \dot{\mathbf{q}}^*), (\dot{\mathbf{c}}, \dot{\mathbf{q}}^*)\rangle &= \langle\mathbb{M}^r(\mathbf{c}), \dot{\mathbf{q}}^*, \dot{\mathbf{q}}^*\rangle\\ &\quad + \langle\mathbb{M}^d(\mathbf{c}), \dot{\mathbf{c}}, \dot{\mathbf{c}}\rangle + 2\langle\mathbb{N}(\mathbf{c}), \dot{\mathbf{q}}^*, \dot{\mathbf{c}}\rangle.\end{aligned}$$

If E is a Banach space, we denote $\mathcal{L}_2(E \times E)$ the Banach space of the bilinear forms on $E \times E$. Technical estimates together with the regularity results of Section II-H allow us to prove:

Proposition 3: The mappings $\mathbf{c} \in \mathcal{B} \mapsto \mathbb{M}^r(\mathbf{c}) \in \mathcal{L}_2(\mathbf{R}^3 \times \mathbf{R}^3)$, $\mathbf{c} \in \mathcal{B} \mapsto \mathbb{N}(\mathbf{c}) \in \mathcal{L}_2(\mathbf{R}^3 \times \mathcal{S})$ and $\mathbf{c} \in \mathcal{B} \mapsto \mathbb{M}^d(\mathbf{c}) \in \mathcal{L}_2(\mathcal{S} \times \mathcal{S})$ are polynomial. We deduce that the mapping $\mathbf{c} \in \mathcal{B} \mapsto \mathbb{M}(\mathbf{c}) \in \mathcal{L}_2((\mathbf{R}^3 \times \mathcal{S}) \times (\mathbf{R}^3 \times \mathcal{S}))$ has the same regularity.

The elementary potentials and next the entries of the mass matrices can be explicitly computed. It is worth observing that once all of these computations have been done:

Proposition 4: When the shape variable $\mathbf{c}$ belongs to $\mathcal{S}_N$ for some integer $N \geq 1$, then all of the entries M^d_{lj} of $\mathbb{M}^d(\mathbf{c})$ are null for $l > N$ and $j > N$ and likewise, all of the entries $N_{j,l}$ of $\mathbb{N}(\mathbf{c})$ are null for $j > N$. Further, the remaining non-zero elements as well as the entries of $\mathbb{M}^r(\mathbf{c})$ make sense even if $\mathbf{c} \notin \mathcal{B}$ and are polynomial functions in $\mathbf{c}$.

It should be noted that when $\mathbf{c} \notin \mathcal{B}$, the mapping $\phi(\mathbf{c})$ is certainly not a conformal mapping any longer (it is not injective) and the domain $\mathcal{F}^*$ is ill-defined (actually it overlaps itself). Obviously in this case, the elementary potentials do not make sense. However, when $\mathbf{c}$ has only a finite number of non-zero elements, the expressions of the entries of $\mathbb{M}^d(\mathbf{c})$ and $\mathbb{N}(\mathbf{c})$ explicitly computed in term of a_k and b_k $(1 \leq k \leq N)$ keep making sense. This leads us to define:

Definition 2 (Mathematically allowable control): Any continuous, piecewise $\mathcal{C}^1$ function $t \in [0, T] \mapsto \mathbf{c}(t) \in \mathcal{S}_N$ $(N \in \mathbf{N}, N \geq 1)$ for which (i) there exists $\mu \in]0, 1[$ such that $\|\mathbf{c}(t)\|_{\mathcal{T}} = \mu$ for all $t \in]0, T[$ and (ii) identity (7) holds for all t where $\dot{\mathbf{c}}$ exists, is said to be mathematically allowable.

J. Equations of motion

We introduce the so-called *impulses* (they both can be identified with vectors of $\mathbf{R}^3$):

$$\begin{bmatrix}\mathbf{P}\\ \Pi\end{bmatrix} := \mathbb{M}^r(\mathbf{c})\begin{bmatrix}\dot{\mathbf{r}}^*\\ \omega\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\mathbf{L}\\ \Lambda\end{bmatrix} := \langle \mathbb{N}(\mathbf{c}), \dot{\mathbf{c}}\rangle.$$

We compute that for all $\dot{\mathbf{p}} := (\dot{\mathbf{s}}, \widetilde{\omega})^T \in \mathbf{R}^3$:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}\begin{bmatrix}\dot{\mathbf{r}}^*\\ \omega\end{bmatrix}\cdot\dot{\mathbf{p}} - \frac{\partial}{\partial \mathbf{q}}\begin{bmatrix}\dot{\mathbf{r}}^*\\ \omega\end{bmatrix}\cdot\dot{\mathbf{p}} = \begin{bmatrix}\widetilde{\omega}(\dot{\mathbf{r}}^*)^\perp - \omega(\dot{\mathbf{s}}^*)^\perp\\ 0\end{bmatrix}.$$

We next easily obtain that:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\dot{\mathbf{p}} - \frac{\partial L}{\partial \mathbf{q}}\cdot\dot{\mathbf{p}} = \frac{d}{dt}\begin{bmatrix}\mathbf{P}+\mathbf{L}\\ \Pi+\Lambda\end{bmatrix}\cdot\begin{bmatrix}\dot{\mathbf{s}}^*\\ \widetilde{\omega}\end{bmatrix} + \begin{bmatrix}\mathbf{P}+\mathbf{L}\\ \Pi+\Lambda\end{bmatrix}\cdot\begin{bmatrix}\widetilde{\omega}(\dot{\mathbf{r}}^*)^\perp - \omega(\dot{\mathbf{s}}^*)^\perp\\ 0\end{bmatrix}.$$

According to Prop. 3, the Lagrangian function is *smooth* with respect to all its variables, allowing all of the derivatives to be computed. Invoking the *least action principle*, the Euler-Lagrange equations of motion are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\dot{\mathbf{p}} - \frac{\partial L}{\partial \mathbf{q}}\cdot\dot{\mathbf{p}} = 0, \quad \forall\, \dot{\mathbf{p}} \in \mathbf{R}^3.$$

We obtain here the equation:

$$\dot{\mathbf{q}}^* = -(\mathbb{M}^r(\mathbf{c}))^{-1}\langle \mathbb{N}(\mathbf{c}), \dot{\mathbf{c}}\rangle. \tag{11}$$

We can also easily set out directly the equation of motion in terms of $\mathbf{r}$ and θ. To this purpose, we introduce the 3×3 block matrix:

$$\mathcal{R}(\theta) := \begin{bmatrix}R(\theta) & 0\\ 0 & 1\end{bmatrix},$$

and since $\dot{\mathbf{r}}^* = R(\theta)^T\mathbf{r}$, we can rewrite (11) in the form:

$$\dot{\mathbf{q}} = -\mathcal{R}(\theta)(\mathbb{M}^r(\mathbf{c}))^{-1}\langle \mathbb{N}(\mathbf{c}), \dot{\mathbf{c}}\rangle. \tag{12}$$

We can verify that $\det \mathbb{M}^r(\mathbf{c}) \geq \pi^3\rho_0^3/2$ for all $\mathbf{c}$ mathematically of physically allowable. According to Prop. 3 and Prop. 4, we can then state:

Proposition 5: For any function $t \in [0,T] \mapsto \mathbf{c}(t)$ mathematically or physically allowable and any initial condition $\mathbf{q}_0 \in \mathcal{Q}$, the ODE(12) is well-posed and the solution is defined on the whole interval $[0,T]$.

III. CONTROLLABILITY RESULTS

Our main result states that:

Theorem 1: For every $\bar{\mu}$ in $(0,1)$, for every $\varepsilon > 0$, for every reference continuous rigid motion $\bar{\mathbf{q}} : [0,T] \to \mathcal{Q}$ and for any reference continuous shape-changes $\bar{\mathbf{c}} : [0,T] \to \mathcal{E}(\bar{\mu})$, there exists a real number $\mu \in (0,1)$ and an analytic allowable control function $\mathbf{c} : [0,T] \to \mathcal{E}(\mu)$ such that

1) $\|\mathbf{c}(t) - \bar{\mathbf{c}}(t)\|_{\mathcal{S}} \leq \varepsilon$ for all $t \in [0,T]$;
2) The solution $\mathbf{q} : [0,T] \to \mathcal{Q}$ of EDO (12) with initial data $\mathbf{q}(0) = \bar{\mathbf{q}}(0)$ and control function $\mathbf{c}$ satisfies $\|\mathbf{q}(t) - \bar{\mathbf{q}}(t)\|_{\mathcal{Q}} < \varepsilon$, for all $t \in [0,T]$ and $\mathbf{q}(T) = \bar{\mathbf{q}}(T)$.

A. Finite dimensional control problem

The proof of Theorem 1 rests on the use of tools of geometric control theory which apply only to finite dimensional systems. So, let be $N \in \mathbf{N}$, $N \geq 1$ and consider a finite set $J := \{1,2,\ldots,n\}$ and a family $\mathfrak{X}_N := (\mathbf{X}^j)_{j\in J}$ of Lipschitz continuous vector fields on $\mathcal{S}$ where for any $\mathbf{c} \in \mathcal{S}$, $\mathbf{X}^j(\mathbf{c}) := (X^j_k(\mathbf{c}))_{k\geq 1}$, $X^j_k(\mathbf{c}) = x^j_k(\mathbf{c}) + iy^j_k(\mathbf{c})$, $x^j_k(\mathbf{c})$, $y^j_k(\mathbf{c}) \in \mathbf{R}$ ($j \in J$, $k \geq 1$) such that $X^j_k = 0$ if $k > N$ and for all $\mathbf{c} \in \mathcal{S}$:

$$\sum_{k=1}^{N} k(x_k(\mathbf{c})a_k + y_k(\mathbf{c})b_k) = 0, \tag{13a}$$

$$\sum_{k=1}^{N} (x_k(\mathbf{c})b_k - y_k(\mathbf{c})a_k)/(k+1) = 0. \tag{13b}$$

Let $\mathbf{c}_0 \in \mathcal{S}_N$ be such that $\|\mathbf{c}_0\|_{\mathcal{T}} < 1$ and consider a set of piecewise constant functions $\alpha_j : [0,T] \to \mathbf{R}$. Then any solution of the Cauchy problem $\dot{\mathbf{c}} = \sum_{j\in J}\alpha_j X_j$ ($t \in]0,T[$) and $\mathbf{c}(0) = \mathbf{c}_0$ is mathematically allowable. Of course $\mathcal{S}_N$ turns out to be identified with $\mathbf{C}^N$ and we are dealing with a finite dimensional problem that can be rewritten:

$$\begin{bmatrix}\dot{\mathbf{q}}\\ \dot{\mathbf{c}}\end{bmatrix} = \sum_{j=1}^{4}\alpha_j(t)\mathbf{Y}^j(\mathbf{q},\mathbf{c}), \tag{14}$$

where we have set:

$$\mathbf{Y}^j(\mathbf{q},\mathbf{c}) := \begin{bmatrix}-\mathcal{R}(\theta)(\mathbb{M}^r(\mathbf{c}))^{-1}\langle \mathbb{N}(\mathbf{c}), \mathbf{X}^j(\mathbf{c})\rangle\\ \mathbf{X}^j(\mathbf{c})\end{bmatrix}.$$

We denote $\mathfrak{Y}_N := (\mathbf{Y}^j)_{j\in J}$ and for all $\mu \in (0,1)$ and all $N \in \mathbf{N}$, $N \geq 1$, we define $\mathcal{E}_N(\mu) = \{\mathbf{c} \in \mathcal{S}_N : \|\mathbf{c}\|_{\mathcal{T}} = \mu\}$. If we identify $\mathcal{S}_N$ now with $\mathbf{R}^{2N}$ then $\mathcal{E}_N(\mu)$ can be identified with the surface of an ellipsoid which is an analytic manifold of dimension $2N-1$ in $\mathbf{R}^{2N}$ and $\mathfrak{X}_N$ can be seen as a family of vector fields on $\mathcal{E}_N(\mu)$. Likewise, $\mathfrak{Y}_N$ is a family of vector fields on the analytic manifold $\mathcal{Q}\times\mathcal{E}_N(\mu)$. Let us state a finite dimensional version of Theorem 1:

Theorem 2: For all but maybe a finite number of values of ρ_0/ρ_f, every $N \in \mathbf{N}$ ($N \geq 2$), every μ in $(0,1)$, every $\varepsilon > 0$, every reference continuous rigid motion $\bar{\mathbf{q}} : [0,T] \to \mathcal{Q}$ and every reference continuous shape-changes $\bar{\mathbf{c}} : [0,T] \to \mathcal{E}_N(\mu)$, there exists an allowable analytic control function $\mathbf{c} : [0,T] \to \mathcal{E}_N(\mu)$ such that

1) $\|\mathbf{c}(t) - \bar{\mathbf{c}}(t)\|_{\mathcal{S}_N} \leq \varepsilon$ for all $t \in [0,T]$;
2) $\mathbf{c}(0) = \bar{\mathbf{c}}(0)$ and $\mathbf{c}(T) = \bar{\mathbf{c}}(T)$;
3) The solution $\mathbf{q} : [0,T] \to \mathcal{Q}$ of EDO (12) with initial data $\mathbf{q}(0) = \bar{\mathbf{q}}(0)$ and control function $\mathbf{c}$ satisfies $\|\mathbf{q}(t) - \bar{\mathbf{q}}(t)\|_{\mathcal{Q}} < \varepsilon$, for all $t \in [0,T]$ and $\mathbf{q}(T) = \bar{\mathbf{q}}(T)$.

Observe that the reference curve $\bar{\mathbf{c}} : [0,T] \to \mathcal{E}_N(\mu)$ is not required to be allowable.

B. Proof of Theorem 2

Recall that, as a classical consequence of the Orbit Theorem, it is enough to find a family $(\mathbf{X}^j)_{j\in J}$ of vector fields whose integral curves are admissible and for which $\mathrm{Lie}_{(\mathbf{q},\mathbf{c})}(\mathbf{Y}^j, j \in J) = T_{(\mathbf{q},\mathbf{c})}(\mathcal{Q}\times\mathcal{E}_N(\mu))$ for every $(\mathbf{q},\mathbf{c})$ to prove Theorem 2.

We specify $J = \{1,2,3,4\}$ and $N = 2$ and we claim:

Proposition 6: There exists a family of analytic vector fields $\mathfrak{X}_2$ on $\mathcal{S}$ such that, for all $\mu \in (0,1)$, $\mathfrak{X}_2$, seen as a vector field on $\mathcal{E}_2(\mu)$, be completely nonholonomic , i.e., for any $\mathbf{c}$ in $\mathcal{E}_2(\mu)$, $\mathrm{Lie}_{\mathbf{c}}(\mathfrak{X}_2) = T_{\mathbf{c}}\mathcal{E}_2(\mu)$.

Proof: The vector fields can be produced explicitly, the entries x_k^j and y_k^j being polynomial functions in a_k and b_k and next the computations of the Lie brackets can also be done explicitly. ■

Proposition 7: For all but maybe a finite number of ρ_0/ρ_f and for any $\mu \in (0,1)$, $\mathfrak{Y}_2$ seen as a vector field on $\mathcal{Q}\times\mathcal{E}_2(\mu)$ is completely nonholonomic.

Proof: Since we are able to produce the expressions of the vector fields of $\mathfrak{X}_2$ as well as the entries of the matrices $\mathbb{M}^r(\mathbf{c})$ and $\mathbb{N}(\mathbf{c})$, we can also compute explicitly the expressions of the vector fields of $\mathfrak{Y}_2$. They depend only on θ, $\mathbf{c}$ and the ratio ρ_0/ρ_f. However, these expressions are far to complicated to allow general computations. We use the particular form of ODE (14). We denote Π_3 the projection over the third component in $\mathcal{Q}$, we define the set of vector fields $\widehat{\mathfrak{X}}_2 := (\widehat{\mathbf{X}}^j)_{j\in J}$ where

$$\widehat{\mathbf{X}}^j(\mathbf{c}) := \begin{bmatrix} -\Pi_3(\mathbb{M}^r(\mathbf{c})^{-1}\langle \mathbb{N}(\mathbf{c}), \mathbf{X}^j(\mathbf{c})\rangle) \\ \mathbf{X}^j(\mathbf{c}) \end{bmatrix},$$

and we observe that:

$$\begin{bmatrix} \dot\theta \\ \dot{\mathbf{c}} \end{bmatrix} = \sum_{j=1}^{N} \alpha_j(t)\widehat{\mathbf{X}}^j(\mathbf{c}). \tag{15}$$

Although not depending on θ, the vector fields $\widehat{\mathbf{X}}^j$ are defined on the analytic manifold $\widehat{\mathcal{E}}_2(\mu) := \mathbf{R}/2\pi \times \mathcal{E}_2(\mu)$. The expressions of the vector fields are still too complicated to allow explicit computations of the Lie brackets in general, however, these computations can be done for a specific value $\mathbf{c} := \mathbf{c}^\dagger$. We obtain that for any θ (since the quantities do not depend on θ) $\mathrm{Lie}_{(\theta,\mathbf{c}^\dagger)}(\widehat{\mathfrak{X}}_2) = T_{(\theta,\mathbf{c}^\dagger)}\widehat{\mathcal{E}}_2(\mu)$. But, according to Prop. 6, any point $(\theta,\mathbf{c}) \in \widehat{\mathcal{E}}_2(\mu)$ is on the orbit of a point $(\tilde\theta, \mathbf{c}^\dagger)$ for some $\tilde\theta \in \mathbf{R}/2\pi$ which entails:

Lemma 1: The family of analytic vector fields $\widehat{\mathfrak{X}}_2$ on $\widehat{\mathcal{E}}_2(\mu)$ is completely nonholonomic.

For a specific couple $(\theta^\dagger, \mathbf{c}^\dagger)$ and a specific value of ρ_0/ρ_f for which we are able to compute explicitly $\mathrm{Lie}_{(\mathbf{r},\theta^\dagger,\mathbf{c}^\dagger)}(\mathfrak{Y}_2)$ (remember that the vector fields in $\mathfrak{Y}_2$ do not depend on $\mathbf{r}$), we obtain that $\mathrm{Lie}_{(\mathbf{r},\theta^\dagger,\mathbf{c}^\dagger)}(\mathfrak{Y}_2) = T_{(\mathbf{r},\theta^\dagger,\mathbf{c}^\dagger)}(\mathcal{Q}\times\mathcal{E}_2(\mu))$ for any $\mu \in (0,1)$ (the expression of $\mathbf{Y}^j$ does not involve μ). We deduce that the conclusion still holds true for all but a finite number of ratios ρ_0/ρ_f by invoking an argument of analyticity and we next conclude as in the proof of Lemma 1 above. ■

We can generalize these results to any $N \geq 2$. The fundamental point is that for any $N > 2$ (and any $\mu \in (0,1)$), $\mathcal{E}_2(\mu)$ can be seen as an immersed submanifold of $\mathcal{E}_N(\mu)$ and likewise, $\mathfrak{X}_2$ can be seen as a complete distribution on $\mathcal{E}_N(\mu)$. We construct (explicitly) a sequence of analytic distributions $\mathfrak{X}_k$ on $\mathcal{E}_k(\mu)$ ($k = 3,\ldots,N$) such that $\mathfrak{X}_{k-1} \subset \mathfrak{X}_k$ and $\mathrm{Lie}_{\mathbf{c}}(\mathfrak{X}_k) = T_{\mathbf{c}}\mathcal{E}_k(\mu)$ for all $\mathbf{c} \in \mathcal{E}_k(\mu)$. We next consider the associated distributions $\widehat{\mathfrak{X}}_k$ and $\mathfrak{Y}_k$ and we check that

$$\mathrm{Lie}_{(\theta,\mathbf{c}^\dagger)}(\widehat{\mathfrak{X}}_2) \oplus \mathrm{span}(\widehat{\mathfrak{X}}_N) = T_{(\theta,\mathbf{c}^\dagger)}\widehat{\mathcal{E}}_N(\mu),$$

where $\mathbf{c}^\dagger$ matches that which we have chosen for the case $N = 2$ (so we can reuse the already done computations for this case). Likewise, choosing the same values for $(\theta^\dagger, \mathbf{c}^\dagger)$ and ρ_0/ρ_f as for the proof of Theorem 2, we show that:

$$\mathrm{Lie}_{(\mathbf{r},\theta^\dagger,\mathbf{c}^\dagger)}(\mathfrak{Y}_2) \oplus \mathrm{span}(\mathfrak{Y}_N) = T_{(\mathbf{r},\theta^\dagger,\mathbf{c}^\dagger)}(\mathcal{Q}\times\mathcal{E}_N(\mu)),$$

and once more, we have no additional intricate Lie brackets to compute! We have just proved the existence of a set of piecewise constant functions $(\alpha_j)_{j\in J}$ such that the solution of System (14) tracks our reference data $\bar{\mathbf{c}}$ and $\bar{\mathbf{q}}$. Finally, the proof of Theorem 2 is completed after adding that the analytic real functions are dense for the L^1 norm in the set of measurable bounded functions and hence that the piecewise constant control functions $(\alpha_j)_{j\in J}$ we have obtained can be approximated by a suitable family of analytic functions.

C. Proof of Theorem 1

Let ε, $\bar\mu$ and $\bar{\mathbf{c}} : [0,T] \to \mathcal{E}(\bar\mu)$ be given as in the statement of Theorem 1 and for any N in $\mathbf{N}$, define $\Theta_N = \{t \in [0,T] : \|\bar{\mathbf{c}}(t) - \Pi_N\bar{\mathbf{c}}(t)\|_{\mathcal{S}} < \varepsilon/4\}$. Because $\bar{\mathbf{c}}$ is continuous, the set Θ_N is open in $[0,T]$ for all N and since for any $t \in [0,T]$, $\Pi_N\bar{\mathbf{c}}(t) \to \bar{\mathbf{c}}(t)$ as $N \to \infty$, we deduce that $[0,T] \subset \cup_{N\geq 1}\Theta_N$. The interval $[0,T]$ being compact and the sequence $(\Theta_N)_N$ non-decreasing, $[0,T] \subset \Theta_N$ for some N. We can not yet choose $\Pi_N\bar{\mathbf{c}}$ as a *good* finite dimensional approximation of $\bar{\mathbf{c}}$ since $\|\Pi_N\bar{\mathbf{c}}(t)\|_{\mathcal{T}}$ is certainly not constant (note that this quantity is continuous in t since $\|\mathbf{c}\|_{\mathcal{T}} \leq \|\mathbf{c}\|_{\mathcal{S}}$ for all $\mathbf{c} \in \mathcal{S}$). We only have to renormalize it. Indeed, we can find a good μ such that $\|\mu\Pi_N\bar{\mathbf{c}}(t)/\|\Pi_N\bar{\mathbf{c}}(t)\|_{\mathcal{T}} - \bar{\mathbf{c}}(t)\|_{\mathcal{T}} < \varepsilon/4$ and it remains to set $\widetilde{\mathbf{c}}(t) := \mu\Pi_N\bar{\mathbf{c}}(t)/\|\Pi_N\bar{\mathbf{c}}(t)\|_{\mathcal{T}}$ to get a continuous function valued in $\mathcal{E}_N(\mu)$ and such that $\|\bar{\mathbf{c}}(t) - \widetilde{\mathbf{c}}(t)\|_{\mathcal{S}} < \varepsilon/2$.

Apply now Theorem 2 with $\widetilde{\mathbf{c}}$ as reference curve and $\varepsilon/2$ instead of ε to conclude the proof.

IV. NUMERICAL RESULTS

This Section is to be read with a web page containing further explanations, all of the animations and many other numerical experiments. It is located at: `http://www.iecn.u-nancy.fr/~munnier/page_amoeba/control_index.html`.

We first choose $N = 2$ (i.e. the shape variable $t \mapsto \mathbf{c}(t)$ has only its two first components $c_k(t) = a_k(t) + ib_k(t)$, $k = 1,2$ as non-zero elements). To manage the constraint that the shape variable has to be allowable, we define it by: $a_k(t) := R_k(t)\cos(\beta_k(t))$ and $b_k(t) := R_k(t)\sin(\beta_k(t))$, where $R_1(t) := \mu\cos(\alpha(t))$, $R_2(t) := \mu\sin(\alpha(t))/\sqrt{2}$ and $\beta_1(t) := -(1/3)\int_0^t h(s)R_2^2(s)\mathrm{d}s$, $\beta_2(t) := (1/2)\int_0^t h(s)R_1^2(s)\mathrm{d}s$. Our new control variables are now the couple $t \in \mathbf{R}_+ \mapsto (\alpha(t), h(t)) \in \mathbf{R}^2$ and for any couple of such functions, the relating control variable $\mathbf{c}$ is allowable. We observe that the function $t \in \mathbf{R}_+ \mapsto \alpha(t) \in \mathbf{R}$ governs the frequencies of the strokes while $t \in \mathbf{R}_+ \mapsto$

$h(t) \in \mathbf{R}$ allows the swimming body to steer left and right. We specialize $\rho_f = 1$, $\rho_0 = (1-\mu^2)\rho_f$ (neutrally buoyant case) with $\mu = 1/2$.

In Fig. 1, we display some screenshots of a motion obtained with $\alpha(t) = t$ and $h(t) = 0$. By specifying next

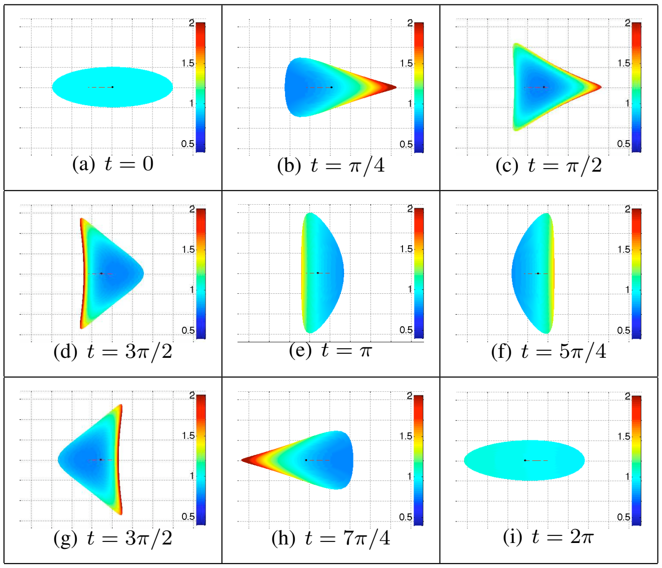

Fig. 1. Screenshots of the amoeba during the course of a stroke. The colors give the value of the internal density. The animal is neutrally buoyant, so at rest its density is 1 (the density of the fluid).

$\alpha(t) = t$ and $h(t) = 1$, we obtain a circular motion as shown in Fig. 2. Suitable choices for α and h allows the amoeba to

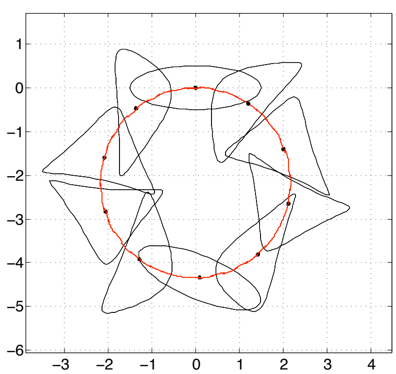

Fig. 2. Successive positions and shapes of the amoeba in its course when $h = 1$. The animal follows a circular trajectory completed over a time interval of length approximately 24π.

follow any smooth trajectory. Such examples as well as a real time interactive game (in Matlab) and *Moonwalk* animations are given on the web page referenced at the beginning of this Section.

V. Conclusion

Through our model of swimming amoeba, we have proved that locomotion in a prefect fluid by shape-changes is theoretically possible. More general 2D swimming strategies as well as 3D models remain to be investigated.

References

[1] F. Alouges, A. DeSimone, and A. Lefebvre. Optimal strokes for low Reynolds number swimmers: an example. *J. Nonlinear Sci.*, 18(3):277–302, 2008.

[2] J. Carling, T. Williams, and G. Bowtell. Self-propelled anguilliform swimming: simultaneous solution of the two-dimentional navier-stokes equations and newton's laws of motion. *J. of Experimental Biology*, 201:3143–3166, 1998.

[3] T. Chambrion and A. Munnier. On the locomotion and control of a self-propelled shape-changing body in a fluid. Submitted to *J. of Nonlinear Science*, 2010.

[4] T. Chambrion and M. Sigalotti. Tracking control for an ellipsoidal submarine driven by Kirchhoff's laws. *IEEE Trans. Automat. Control*, 53(1):339–349, 2008.

[5] A. Cherman, J. Delgado, F. Duda, K. Ehlers, J. Koiller, and R. Montgomery. Low Reynolds number swimming in two dimensions. In *Hamiltonian systems and celestial mechanics (Pátzcuaro, 1998)*, volume 6 of *World Sci. Monogr. Ser. Math.*, pages 32–62. World Sci. Publ., River Edge, NJ, 2000.

[6] S. Childress. *Mechanics of swimming and flying*, volume 2 of *Cambridge Studies in Mathematical Biology*. Cambridge University Press, Cambridge, 1981.

[7] G. P. Galdi. On the steady self-propelled motion of a body in a viscous incompressible fluid. *Arch. Ration. Mech. Anal.*, 148(1):53–88, 1999.

[8] E. Kanso, J. E. Marsden, C. W. Rowley, and J. B. Melli-Huber. Locomotion of articulated bodies in a perfect fluid. *J. Nonlinear Sci.*, 15(4):255–289, 2005.

[9] S. D. Kelly and R. M. Murray. Modelling efficient pisciform swimming for control. *Internat. J. Robust Nonlinear Control*, 10(4):217–241, 2000.

[10] V. V. Kozlov and D. A. Onishchenko. Motion of a body with undeformable shell and variable mass geometry in an unbounded perfect fluid. *J. Dynam. Differential Equations*, 15(2-3):553–570, 2003.

[11] H. Lamb. *Hydrodynamics*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, sixth edition, 1993.

[12] J. Lighthill. *Mathematical biofluiddynamics*. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1975.

[13] H. Liu and K. Kawachi. A numerical study of undulatory swimming. *J. comput. phys.*, 155(2):223–247, 1999.

[14] R. Mason. *Fluid Locomotion and Trajectory Planning for Shape-Changing Robots*. Ph.d. thesis, California Institute of Technology, Department of Mechanical Engineering, June 2003.

[15] J. B. Melli, C. W. Rowley, and D. S. Rufat. Motion planning for an articulated body in a perfect planar fluid. *SIAM J. Appl. Dyn. Syst.*, 5(4):650–669 (electronic), 2006.

[16] A. Munnier. On the self-displacement of deformable bodies in a potential fluid flow. *Math. Models Methods Appl. Sci.*, 18(11):1945–1981, december 2008.

[17] A. Munnier. Locomotion of deformable bodies in an ideal fluid: Newtonian versus lagrangian formalism. *J. Nonlinear Sci.*, 19(6):665–715, 2009.

[18] J. San Martin, J. F. Scheid, T. Takahashi, and M. Tucsnak. An initial and boundary problem modeling fish-like swimming. *Arch. Ration. Mech. Anal.*, 2008.

[19] J. San Martín, T. Takahashi, and M. Tucsnak. A control theoretic approach to the swimming of microscopic organisms. *Quart. Appl. Math.*, 65(3):405–424, 2007.

[20] A. Shapere and F. Wilczek. Geometry of self-propulsion at low Reynolds number. *J. Fluid Mech.*, 198:557–585, 1989.

[21] J. A. Sparenberg. Survey of the mathematical theory of fish locomotion. *J. Engrg. Math.*, 44(4):395–448, 2002.

[22] G. Taylor. Analysis of the swimming of microscopic organisms. *Proc. R. Soc. Lond., Ser. A*, 209:447–461, 1951.

[23] M. S. Triantafyllou, G. S. Triantafyllou, and D. K. P. Yue. Hydrodynamics of fishlike swimming. In *Annual review of fluid mechanics*, volume 32 of *Annu. Rev. Fluid Mech.*, pages 33–53. Annual Reviews, Palo Alto, CA, 2000.

[24] T. Y. Wu. Mathematical biofluiddynamics and mechanophysiology of fish locomotion. *Math. Methods Appl. Sci.*, 24(17-18):1541–1564, 2001.

[25] Q. Zhu, M. J. Wolfgang, D. K. P. Yue, and M. S. Triantafyllou. Three-dimensional flow structures and vorticity control in fish-like swimming. *J. Fluid Mech.*, 468:1–28, 2002.